Bitte setzt euch in den vordersten vier Reihen!

Lineare Algebra

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Programm

- Theorie-Input
- In-class Exercise
- Nachbesprechung Serie 7

Theorie

Orthogonalität

Lemma 5.1.8. *Let V be a subspace of $\mathbb{R}^n$. Then $V = (V^\perp)^\perp$.*

Corollary 5.1.9. *Let $A \in \mathbb{R}^{m\times n}$. $N(A) = C(A^T)^\perp$ and $C(A^T) = N(A)^\perp$.*

Lemma 5.1.10. *Let $A \in \mathbb{R}^{m\times n}$. Then $N(A) = N(A^TA)$ and $C(A^T) = C(A^TA)$.*

Projektionen

Definition 5.2.1 (Projection of a vector onto a subspace). *The projection of a vector $b \in \mathbb{R}^m$ on a subspace S (of $\mathbb{R}^m$) is the point in S that is closest to b. In other words*

$$\mathrm{proj}_S(b) = \underset{p \in S}{\mathrm{argmin}} \|b - p\|. \tag{1}$$

If a mouse:

gets run over by a steamroller:

Projektionen

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$$\text{proj}_S(b) = \underset{p \in S}{\text{argmin}} \, \|b - p\|. \qquad (1)$$

If a mouse:

It will look like this:

Projektionen auf eine Linie

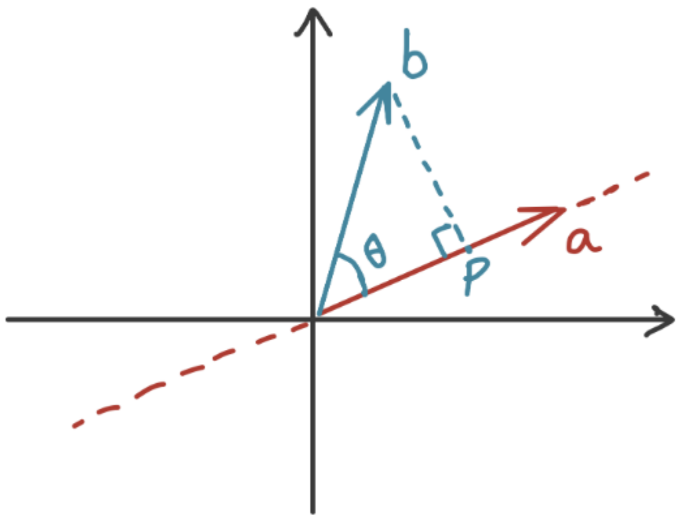

FIGURE 1. Projection on a line.

Lemma 5.2.2. *Let $a \in \mathbb{R}^m \setminus \{0\}$. The projection of $b \in \mathbb{R}^m$ on $S = \{\lambda a \mid \lambda \in \mathbb{R}\} = C(a)$ is given by*

$$\text{proj}_S(b) = \frac{aa^T}{a^T a} b.$$

Projektionen im Allgemeinen

Lemma 5.2.3. *The projection of a vector $b \in \mathbb{R}^m$ to the subspace $S = C(A)$ is well defined. It can be written as*

$$\text{proj}_S(b) = A\hat{x}, \text{ where } \hat{x} \text{ satisfies the normal equations } A^T A\hat{x} = A^T b.$$

$$p = A\hat{x} = A\left(A^\top A\right)^{-1} A^\top b.$$

Lemma 5.2.4. *$A^\top A$ is invertible if and only if A has linearly independent columns.*

Projektionen im Allgemeinen

Lemma 5.2.4. *$A^\top A$ is invertible if and only if A has linearly independent columns.*

- A muss nicht quadratisch sein!

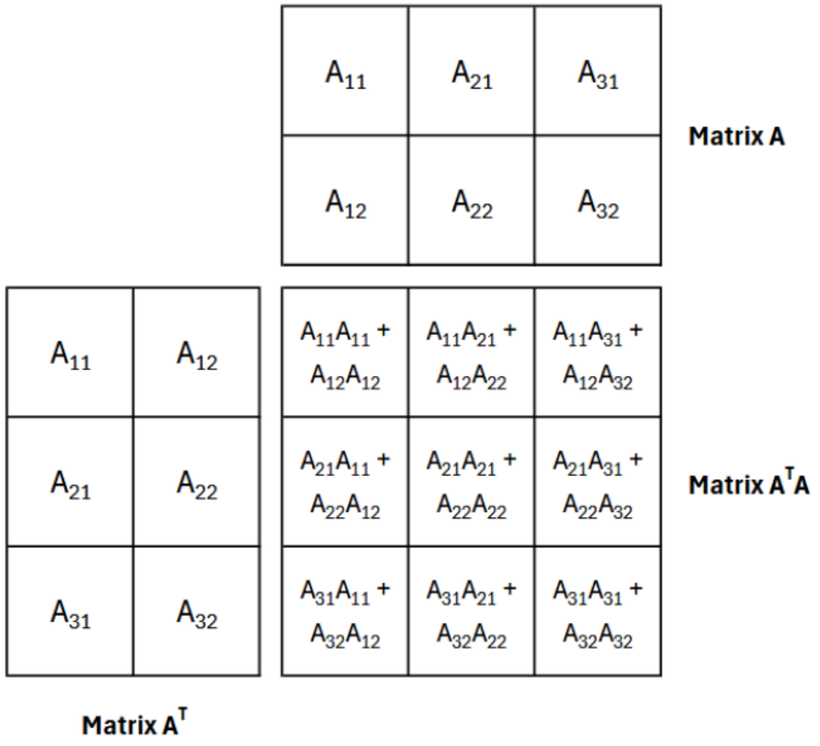

Figure 1: Matrix multiplication $A^\top A$

Projektionen im Allgemeinen

Theorem 5.2.5. *Let S be a subspace in $\mathbb{R}^m$ and A a matrix whose columns are a basis of S. The projection of $b \in \mathbb{R}^m$ to S is given by*

$$\operatorname{proj}_S(b) = Pb,$$

where $P = A\left(A^\top A\right)^{-1} A^\top$ is the projection matrix.

Methode der kleinsten Quadrate (least squares)

- Problem: Wir haben ein Gleichungssystem mit zu vielen Gleichungen:
- $3x + 1 = 4$
- $2x - 1 = 3$
- Wir haben also keine Lösung für das Gleichungssystem!
- Wir wollen eine möglichst gute Lösung finden

Methode der kleinsten Quadrate

- Sei: $A \in \mathbb{R}^{mxn}$ und $b \in \mathbb{R}^{m}$. Wir betrachten das Gleichungssystem $Ax = b$, das keine exakte Lösung hat.
- Eine möglichst gute Lösung ist also definiert durch $\min_{\hat{x} \in \mathbb{R}^n} \|A\hat{x} - b\|^2.$
- Wir wollen also den Unterschied von Ax und b minimieren.

Least Squares und Projektionen

The link to projections

Consider the subspace $C(A) = \{Ax \mid x \in \mathbb{R}^n\}$. Then,

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 = \min_{p \in C(A)} \|b - p\|^2 = \|b - \text{proj}_{C(A)}(b)\|^2.$$

Remark

- A least squares solution is given by $\text{proj}_{C(A)}(b) = A\hat{x}$, where $A^T A\hat{x} = A^T b$.
- If A has linearly independent columns, then $A^T A$ is invertible. Hence, for the least squares solution we have the explicit formula

$$\hat{x} = (A^T A)^{-1} A^T b.$$

Lineare Regression

- Wir betrachten Punkte in $\mathbb{R}^2$: $(t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m)$
- Wir suchen eine Linie, die diese Punkte möglichst gut approximiert:

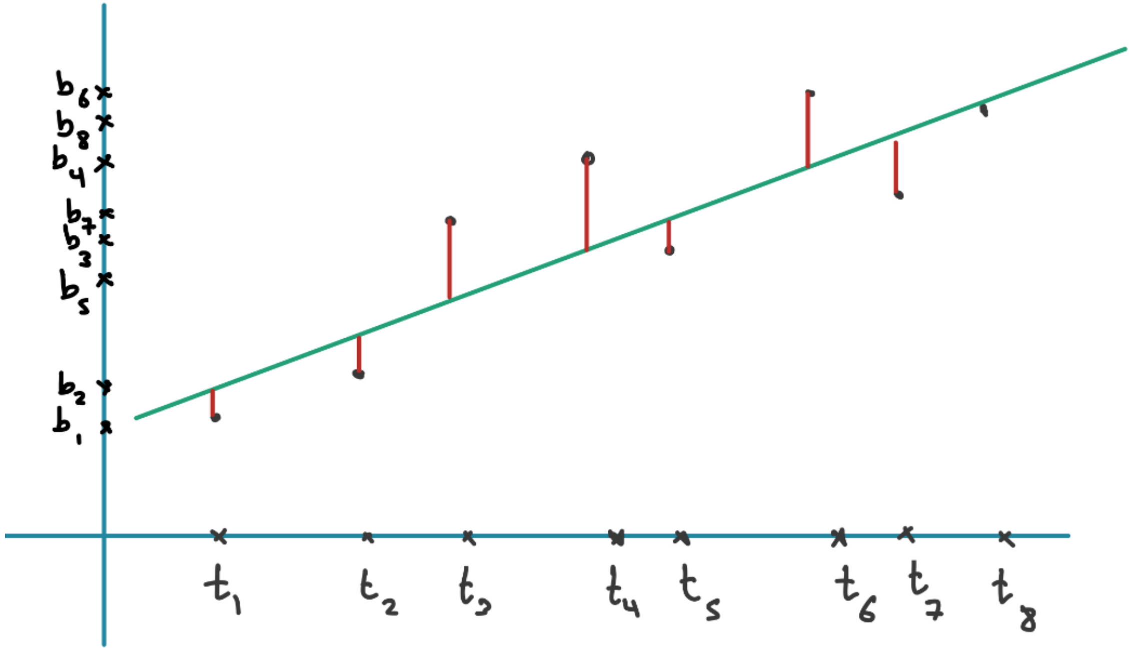

FIGURE 2. Fitting a line to points

Lineare Regression

- Wir betrachten Punkte in $\mathbb{R}^2$: $(t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m)$
- Wir suchen eine Linie, die diese Punkte möglichst gut approximiert, also suchen wir:
- $\alpha_0 \in \mathbb{R}$ and $\alpha_1 \in \mathbb{R}$
- So dass: $b_k \approx \alpha_0 + \alpha_1 t_k.$

Lineare Regression

- Wir suchen also:

In matrix-vector notation

$$\min_{\alpha_0,\alpha_1} \left\| b - A \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \right\|^2, \tag{1}$$

where

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \\ b_m \end{pmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_{m-1} \\ 1 & t_m \end{bmatrix}.$$

We can assume that A has independent columns to get

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = (A^\top A)^{-1} A^\top b = \begin{bmatrix} m & \sum_{k=1}^m t_k \\ \sum_{k=1}^m t_k & \sum_{k=1}^m t_k^2 \end{bmatrix}^{-1} \begin{pmatrix} \sum_{k=1}^m b_k \\ \sum_{k=1}^m t_k b_k \end{pmatrix}$$

Fragen?

Übungen

1. Linear regression (in-class) (★☆☆)

We want to determine the parameters of a certain model function from some measured values. Assume that we measured the following values

t_i	1	2	3	4	5
b_i	2	3	5	6	8

where $i \in [5]$. Moreover, assume that we want to model the relationship between t, b by a function f, i.e. $b = f(t)$. In this exercise, we restrict f to be a line, i.e. f should have the form

$$f(t) = \alpha_1 t + \alpha_0$$

for parameters $\alpha_1, \alpha_0 \in \mathbb{R}$. Our goal is to find suitable values for α_1, α_0. As discussed in the lecture, this idea of fitting a line through a set of datapoints is called linear regression.

a) For each datapoint (t_i, b_i) with $i \in [5]$, we get an equation for α_1, α_0 from $f(t_i) = b_i$. Write down the system of linear equations that we get by combining all five equations.

b) Do you expect this system to have any solutions? (Answer this intuitively without actually solving the system).

c) Using the normal equations, find an approximate solution to the system you wrote down.

Euklidische Norm

Definition 1.11 (Euclidean norm). *Let* $\mathbf{v} \in \mathbb{R}^m$. *The Euclidean norm of* $\mathbf{v}$ *is the number*

$$\|\mathbf{v}\| := \boxed{\sqrt{\mathbf{v} \cdot \mathbf{v}}.}$$

Beweisen, dass zwei Mengen gleich sind

- Um zu zeigen, dass $\mathrm{A} = \mathrm{B}$, zeige:
  - $\mathrm{A} \subseteq \mathrm{B}$
  - $\mathrm{B} \subseteq \mathrm{A}$
- Um zu zeigen, dass $A \subseteq B$:
  - Sei $\mathrm{x} \in \mathrm{A}$ beliebig. Wir zeigen, dass dann auch $\mathrm{x} \in \mathrm{B}$ gilt. Dazu …
- Schreibt auf, was ihr macht!

2. Nullspace and column space (hand-in) (★★☆)

Let $\mathbf{v}$ be a *unit vector* (i.e. $\| \mathbf{v} \| = 1$) in $\mathbb{R}^3$. Consider the 3×3 matrices A and P defined by

$$A := \mathbf{v}\mathbf{v}^\top, \qquad P := I_3 - \mathbf{v}\mathbf{v}^\top = I_3 - A$$

where I_3 is the 3×3 identity matrix.

a) Show that $A^2 = A$ and $P^2 = P$.

b) Let $\mathbf{w} \in \mathbb{R}^3$ be orthogonal to $\mathbf{v}$. Prove $A\mathbf{w} = \mathbf{0}$.

c) Now let $\mathbf{w} \in \mathbb{R}^3$ be a vector satisfying $A\mathbf{w} = \mathbf{0}$. Prove $\mathbf{w} \cdot \mathbf{v} = 0$.

d) Based on b) and c), describe the nullspace $\mathbf{N}(A)$.

e) Determine the rank of A. Is A invertible?

2. Nullspace and column space (hand-in) (★★☆)

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$$A := \mathbf{v}\mathbf{v}^\top, \qquad P := I_3 - \mathbf{v}\mathbf{v}^\top = I_3 - A$$

where I_3 is the 3×3 identity matrix.

f) Prove that $\mathbf{C}(A) = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} = \mathsf{Span}(\mathbf{v})$.

g) Also prove that $\mathbf{C}(A) = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$.

h) Use g) to prove $\mathbf{N}(P) = \mathbf{C}(A)$.

i) Finally, prove $\mathbf{C}(P) = \mathbf{N}(A)$.

5. Orthogonal subspaces in $\mathbb{R}^4$ (★★★)

Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$ and $W = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\} \subseteq \mathbb{R}^4$ be subspaces of $\mathbb{R}^4$.

a) Determine a basis of $V^\perp$.

b) Does there exist a basis of W such that no basis element is in $V^\perp$?

c) Does there exist a basis of W such that exactly one basis element is in $V^\perp$?

d) Does there exist a basis of W such that exactly two basis elements are in $V^\perp$?

e) Does there exist a basis of W such that exactly three basis elements are in $V^\perp$?

Justify your answers.