Bitte setzt euch in den vordersten vier Reihen!

Lineare Algebra

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Programm

- Theorie-Input
- In-class Exercise
- Nachbesprechung Serie 6

Theorie

Lineartransformationen zwischen Vektorräumen

Definition 4.26 (Linear transformation between vector spaces). *Let V, W be two vector spaces. A function $T : V \to W$ is called a* linear transformation between vector spaces *if the following linearity axiom holds for all $\mathbf{x}_1, \mathbf{x}_2 \in V$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$.*

$$T(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) = \lambda_1 T(\mathbf{x}_1) + \lambda_2 T(\mathbf{x}_2).$$

Isomorphismen

Lemma 4.27 (Bijective linear transformations preserve bases). *Let $T : V \to W$ be a bijective linear transformation between vector spaces V and W. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\} \subseteq V$ be a finite set of some size ℓ, and $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_\ell)\} \subseteq W$ the transformed set. Then $|T(B)| = |B|$. Moreover, B is a basis of V if and only if $T(B)$ is a basis of W. We therefore also have* $\dim(V) = \dim(W)$.

Definition 4.28 (Isomorphic vector spaces, isomorphism). *Let V, W be two vector spaces. If there is a bijective linear transformation $T : V \to W$ (Definition 4.26), then V and W are called* isomorphic, *and T is called an* isomorphism *between V and W.*

Die drei fundamentalen Unterräume

- Spaltenraum
- Zeilenraum
- Nullraum

Basis vom Spaltenraum

Theorem 4.31: If R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$, then A has its independent columns at indices $j_1, j_2, \ldots, j_r$, and these columns form a basis of the column space $\mathbf{C}(A)$. In particular,

$$\dim(\mathbf{C}(A)) = \mathbf{rank}(A) = r.$$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \quad \rightarrow \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad j_1 \qquad j_2$$

Basis vom Zeilenraum

Theorem 4.32: If R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$, the first r rows of R form a basis of the row space $\mathbf{R}(A)$. In particular,

$$\dim(\mathbf{R}(A)) = r.$$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \quad \to \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Spaltenrang = Zeilenrang

Theorem 4.33. *Let A be an $m \times n$ matrix. Then*

$$\mathbf{rank}(A) = \mathbf{rank}(A^\top).$$

Basis vom Nullraum

Theorem 4.36. *Let A be an $m \times n$ matrix, and let R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$ be the result of Gauss-Jordan elimination on A according to Theorem 3.17. Let $k_1, k_2, \ldots, k_{n-r}$ be the column indices not in $\{j_1, j_2, \ldots, j_r\}$, and let Q be the $r \times (n-r)$ submatrix of R containing the first r rows and columns $k_1, k_2, \ldots, k_{n-r}$.*

A basis of the nullspace $\mathbf{N}(A)$ is given by the $n-r$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-r}$, where $\mathbf{v}_i$ is the vector $\mathbf{x} \in \mathbf{N}(R)$ with

$$\begin{pmatrix} x_{k_1} \\ x_{k_2} \\ \vdots \\ x_{k_{n-r}} \end{pmatrix} = \mathbf{e}_i, \qquad \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \vdots \\ x_{j_r} \end{pmatrix} = -Q\mathbf{e}_i.$$

We note that $Q\mathbf{e}_i$ is the i-th column of Q. In particular,

$$\dim(\mathbf{N}(A)) = n - r.$$

Beispiel

Theorem 4.36. *Let A be an $m \times n$ matrix, and let R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$ be the result of Gauss-Jordan elimination on A according to Theorem 3.17. Let $k_1, k_2, \ldots, k_{n-r}$ be the column indices not in $\{j_1, j_2, \ldots, j_r\}$, and let Q be the $r \times (n-r)$ submatrix of R containing the first r rows and columns $k_1, k_2, \ldots, k_{n-r}$.*

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We note that $Q\mathbf{e}_i$ is the i-th column of Q. In particular,

$$\dim(\mathbf{N}(A)) = n - r.$$

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ 3 & -4 & 1 & 5 \\ -2 & 2 & -1 & -4 \end{bmatrix}$$

Lösungsraum von Ax = b

Definition 4.37: Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. The set

$$\mathbf{Sol}(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} \subseteq \mathbb{R}^n$$

is the *solution space* of $A\mathbf{x} = \mathbf{b}$.

Lösungsraum berechnen

Theorem 4.38 (Solution space from shifting the nullspace). *Let A be an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{s}$ be some solution of $A\mathbf{x} = \mathbf{b}$. Then*

$$\mathbf{Sol}(A, \mathbf{b}) = \{\mathbf{s} + \mathbf{x} : \mathbf{x} \in \mathbf{N}(A)\}.$$

Hence, we can also compute $\mathbf{Sol}(A, b)$, despite the fact that it is not a subspace. To describe all solutions, we just need *some* solution $\mathbf{s}$ (for example the canonical one from Section 3.3.2) and a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-r}\}$ of $\mathbf{N}(A)$ (see Theorem 4.36). Then

$$\mathbf{Sol}(A, \mathbf{b}) = \left\{ \mathbf{s} + \sum_{i=1}^{n-r} \lambda_i \mathbf{v}_i : \lambda_i \in \mathbb{R} \text{ for } i \in [n - r] \right\}.$$

Orthogonale Vektoren/Vektorräume

Definition 5.1.1. *Two vectors $v, w \in \mathbb{R}^n$ are called orthogonal if $v^T w = \sum_{i=1}^n v_i w_i = 0$. Two subspaces V and W are orthogonal if for all $v \in V$ and $w \in W$, the vectors v and w are orthogonal.*

Lemma 5.1.2. *Let $v_1, \dots, v_k$ be a basis of subspace V. Let $w_1, \dots, w_l$ be a basis of subspace W. V and W are orthogonal if and only if v_i and w_j are orthogonal for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$.*

Orthogonales Komplement

Definition 5.1.5. *Let V be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of V as*

$$V^\perp = \{w \in \mathbb{R}^n \mid w^T v = 0 \textit{ for all } v \in V\}.$$

Das orthogonale Komplement ist ein Unterraum von R^n

Orthogonales Komplement

Theorem 5.1.6. *Let* $A \in \mathbb{R}^{m\times n}$ *be a matrix.*

$$N(A) = C(A^T)^\perp = R(A)^\perp.$$

Orthogonales Komplement

Theorem 5.1.7. *Let V, W be orthogonal subspaces of $\mathbb{R}^n$.*

The following statements are equivalent.

(i) $W = V^\perp$.

(ii) $\dim(V) + \dim(W) = n$.

(iii) *Every $u \in \mathbb{R}^n$ can be written as $u = v + w$ with unique vectors $v \in V$, $w \in W$.*

Fragen?

Übungen

1. Solving linear systems (in-class) (★☆☆)

Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} -1 & 2 & 5 & -2 \\ -3 & 3 & 12 & -3 \\ 1 & -14 & -7 & -6 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} -6 \\ -15 \\ 8 \end{pmatrix}.$$

a) Determine the set of solutions $\mathbf{Sol}(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{b}\}$, i.e., write down an *explicit* characterization of this set of solutions in the form presented after Theorem 4.38 on page 155 in the lecture notes.

b) Write down a basis for $\mathbf{N}(A)$ (you might have already found it in the previous subtask), and also find a basis for $\mathbf{C}(A)$.

c) What are the dimensions of $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$, and $\mathbf{R}(A)$?

d) Determine a basis of $\mathbf{R}(A)$.

3. Skew-symmetric matrices as a subspace (bonus, hand-in) (★★☆)

Let $m \in \mathbb{N}^+$. A matrix $A \in \mathbb{R}^{m \times m}$ is skew-symmetric if and only if $A = -A^\top$. Consider the set $\mathcal{S}_m$ of skew-symmetric $m \times m$ matrices.

a) Show that $\mathcal{S}_m$ is a subspace of $\mathbb{R}^{m \times m}$.

b) What is the dimension of $\mathcal{S}_m$? Justify your answer with a proof.

Hint: You can use Assignment 2 Exercise 6b) without a proof.

6. Union of subspaces (★★☆)

Let V be a vector space and let U and W be subspaces of V. Show that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

2. Strictly diagonally dominant matrices (hand-in) (★★★)

Let $A \in \mathbb{R}^{m \times m}$ be strictly diagonally dominant, that is,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}|$$

holds for all $i \in [m]$. Prove that A is invertible.