Bitte setzt euch in den vordersten vier Reihen!

# Lineare Algebra

Übung 6, 30. Oktober 2025

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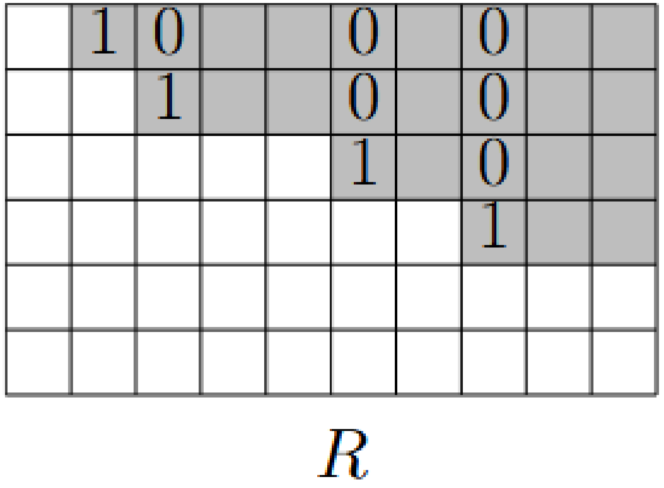
# Programm

- Theorie-Input
- In-class Exercise
- Nachbesprechung Serie 5

# Theorie

# Gauss-Jordan Elimination

- Ziel: Reduzierte Zeilenstufenform




Alle weissen Einträge sind 0

**Definition 3.13** (Reduced row echelon form). *Let* $R = [r_{ij}]_{i=1,j=1}^{m\ \ n}$ *be an* $m \times n$ *matrix.* $R$ *is in* reduced row echelon form (RREF) *if there is some natural number* $r \leq m$ *and column indices* $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ *(the indices of the "downward step" columns) such that the following two conditions hold.*

*(i) For every* $i \in [r]$*, column* $j_i$ *of* $R$ *is the standard unit vector* $\mathbf{e}_i$*.*

*(ii) All entries* $r_{ij}$ *"below the staircase" are* 0*. Formally, an entry* $r_{ij}$ *is* below the staircase *if*

*(a)* $i > r$ *(the entry is below row* $r$*), or*

*(b)* $i \leq r$ *and* $j < j_i$ *(the entry is in the part of row* $i$ *to the left of column* $j_i$*).*

*If we want to describe the shape of* $R$ *precisely, we say that* $R$ *is in* $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$*.*

# Reduzierte Zeilenstufenform

- Wenn wir Rx = c haben, wobei R in reduzierter Zeilenstufenform ist, können wir x einfach ausrechnen

**Algorithm 5** Direct solution:
Returns a pair $(\mathbf{x}, \text{result})$ such that $R\mathbf{x} = \mathbf{c}$ if result = "solution". If result = "no solution", there is no solution. The matrix $R$ must be in $\text{RREF}(j_1, j_2, \ldots, j_r)$.

**function** DIRECT SOLUTION$(R, j_1, j_2, \ldots, j_r, \mathbf{c})$ ▷ $R \in \mathbb{R}^{m \times n}, \mathbf{c} \in \mathbb{R}^m$
    $\mathbf{x} \leftarrow \mathbf{0} \in \mathbb{R}^n$
    **if** $c_i \neq 0$ for some $i > r$ **then**
        **return** $(\mathbf{x}$, "no solution")
    **end if**
    **for** $i = 1, 2, \ldots, r$ **do**
        $x_{j_i} \leftarrow c_i$
    **end for**
    **return** $(\mathbf{x}$, "solution")
**end function**

# Gauss-Jordan Elimination

- Transformiert $Ax = b$ zu $Rx = c$ mit denselben Lösungen, aber R ist in reduzierter Zeilenstufenform
- Gleiche Idee wie Gauss-Elimination, aber:
- Falls wir ein Nullpivot haben, den wir nicht durch Tauschen der Spalten eliminieren können, geben wir nicht auf, sondern gehen einfach zur nächsten Spalte

$$\begin{bmatrix} 1 & 2 & 1 & 1 & -1 \\ 0 & \color{red}{0} & 0 & 1 & 7 \\ 0 & \color{red}{0} & -2 & -2 & 10 \end{bmatrix}$$

# Gauss-Jordan Elimination ausrechnen

- Transformiert $Ax = b$ zu $Rx = c$ mit denselben Lösungen, aber R ist in reduzierter Zeilenstufenform
- Gleiche Idee wie Gauss-Elimination, aber:
- Falls das Pivot-Element nicht 1 ist, dividieren wir die Reihe durch das Pivot-Element:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & -1 \\ 0 & 0 & \mathbf{-2} & -2 & 10 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & -1 \\ 0 & 0 & \mathbf{1} & 1 & -5 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

# Gauss-Jordan Elimination ausrechnen

- Transformiert $Ax = b$ zu $Rx = c$ mit denselben Lösungen, aber R ist in reduzierter Zeilenstufenform
- Gleiche Idee wie Gauss-Elimination, aber:
- Falls die Elemente über dem Pivot-Element nicht null sind, nutzen wir den Pivot, um diese zu eliminieren:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & -1 \\ 0 & 0 & \mathbf{1} & 1 & -5 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & \mathbf{1} & 1 & -5 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

# Beispiel Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & -3 & -5 & -2 \\ 3 & -1 & 1 & 8 \end{bmatrix}$$

# CR-Dekomposition mit Gauss-Jordan

**Theorem 3.18** (Uniqueness of RREF, and relation to the CR decomposition). *Let $A$ be an $m \times n$ matrix. There is a unique $m \times n$ matrix $R$ (the one resulting from Gauss-Jordan elimination on $A$ according to Theorem 3.17), with the following two properties.*

*(i)* $R = MA$ *for some invertible* $m \times m$ *matrix* $M$.

*(ii)* $R$ *is in* RREF.

*More precisely,* $R$ *is in* $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$, *where* $j_1, j_2, \ldots, j_r$ *are the indices of the independent columns in* $A$, *and*

$$R = \left[\begin{array}{c} \underbrace{R'}_{r \times n} \\ \hline \underbrace{0}_{(m-r)\times n} \end{array}\right],$$

*with* $R'$ *the unique matrix such that* $A = CR'$ *in Theorem 2.46 (CR decomposition).*

# Beispiel CR-Dekomposition mit Gauss-Jordan Elimination

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ 3 & -4 & 1 & 5 \\ -2 & 2 & -1 & -4 \end{bmatrix}$$

# Inverse berechnen mit Gauss-Jordan Elimination

**Theorem 3.19** (Computing inverses with Gauss-Jordan elimination). *Let $A$ be an $m \times m$ matrix, and let $(R, j_1, j_2, \ldots, j_r, M)$ be the output of running Algorithm 6 with input $(A, I)$. Then $A$ is invertible if and only if $R = I$, and in this case, $A^{-1} = M$.*

- Idee: Folgende Matrix mit Gauss-Jordan Elimination umformen. Falls linke Seite = I ist, dann ist die rechte Seite die Inverse von A

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

# Beispiel Inverse berechnen mit Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

# Vektorräume

- Ein Vektorraum ist eine Menge von Elementen mit zwei Operationen: Addition und Skalarmultiplikation.
- Die Elemente vom Vektorraum heissen Vektoren
- Wenn man Vektoren addiert ($v + w$) oder skaliert ($\lambda v$), dann sind die Resultate wieder im Vektorraum.

# Vektorräume

**Definition 4.1** (Vector space). *A* vector space *is a triple* $(V, +, \cdot)$ *where* $V$ *is a set (the vectors), and*

$$
\begin{array}{lcll}
+ & : & V \times V \to V & \textit{is a function (vector addition),} \\
\cdot & : & \mathbb{R} \times V \to V & \textit{is a function (scalar multiplication),}
\end{array}
$$

*satisfying the following* axioms of a vector space *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *and all* $\lambda, \mu \in \mathbb{R}$.

| | | |
|---|---|---|
| 1. | $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ | *commutativity* |
| 2. | $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ | *associativity* |
| 3. | *There is a vector* $\mathbf{0}$ *such that* $\mathbf{v} + \mathbf{0} = \mathbf{v}$ *for all* $\mathbf{v}$ | *zero vector* |
| 4. | *There is a vector* $-\mathbf{v}$ *such that* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | *negative vector* |
| 5. | $1 \cdot \mathbf{v} = \mathbf{v}$ | *identity element* |
| 6. | $(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$ | *compatibility of* $\cdot$ *and* $\cdot$ *in* $\mathbb{R}$ |
| 7. | $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$ | *distributivity over* $+$ |
| 8. | $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ | *distributivity over* $+$ *in* $\mathbb{R}$ |

# Beispiele von Vektorräumen

- Vektoren

**Observation 4.2.** $(\mathbb{R}^m, +, \cdot)$, *with "+" as in Definition 1.2 and "$\cdot$" as in Definition 1.3, is a vector space.*

# Beispiele von Vektorräumen

- Polynome

**Theorem 4.4.** *Let* $\mathbb{R}[x]$ *denote the set of polynomials in one variable* $x$. *Given two polynomials* $\mathbf{p} = \sum_{i=0}^{m} p_i x^i$ *and* $\mathbf{q} = \sum_{i=0}^{n} q_i x^i$, *we define* $\mathbf{p} + \mathbf{q}$ *to be the polynomial*

$$\mathbf{p} + \mathbf{q} = \sum_{i=0}^{\max(m,n)} (p_i + q_i) x^i,$$

*where we set* $p_i = 0$ *for* $i > m$ *and* $q_i = 0$ *for* $i > n$. *For a scalar* $\lambda \in \mathbb{R}$, *we further define* $\lambda\mathbf{p}$ *as the polynomial*

$$\lambda\mathbf{p} = \sum_{i=0}^{m} (\lambda p_i) x^i.$$

*Then* $(\mathbb{R}[x], +, \cdot)$ *is a vector space.*

# Beispiele von Vektorräumen

- Matrizen

**Theorem 4.5.** *Let $\mathbb{R}^{m \times n}$ be the set of $m \times n$ matrices, with addition $A + B$ and scalar multiplication $\lambda A$ defined in the usual way, see Definition 2.2. Then $(\mathbb{R}^{m \times n}, +, \cdot)$ is a vector space.*

# Eigenschaften von Vektorräumen

**Fact 4.6.** *Let* $(V, +, \cdot)$ *be a vector space.* $V$ *contains exactly one zero vector (a vector satisfying axiom 3 of Definition 4.1:* $\mathbf{v} + \mathbf{0} = \mathbf{v}$ *for all* $\mathbf{v}$*).*

*Proof.* Take two zero vectors $\mathbf{0}$ and $\mathbf{0}'$. Then

$$\begin{aligned} \mathbf{0}' &= \mathbf{0}' + \mathbf{0} && \text{(by axiom 3, since } \mathbf{0} \text{ is a zero vector)} \\ &= \mathbf{0} + \mathbf{0}' && \text{(by axiom 1, commutativity)} \\ &= \mathbf{0} && \text{(by axiom 3, since } \mathbf{0}' \text{ is a zero vector).} \end{aligned}$$

So $\mathbf{0}$ and $\mathbf{0}'$ are equal. □

# Eigenschaften von Vektorräumen

**Lemma**

*For every vector $\mathbf{v}$, we have $0 \cdot \mathbf{v} = \mathbf{0}$.*

**Lemma**

*Each $\mathbf{v}$ has only one negative vector.*

# Eigenschaften von Vektorräumen

**Lemma**

*For every vector* $\mathbf{v}$*, we have* $0 \cdot \mathbf{v} = \mathbf{0}$.

Vector space axioms:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$
4. There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $1 \cdot \mathbf{v} = \mathbf{v}$
6. $(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$
7. $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
8. $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

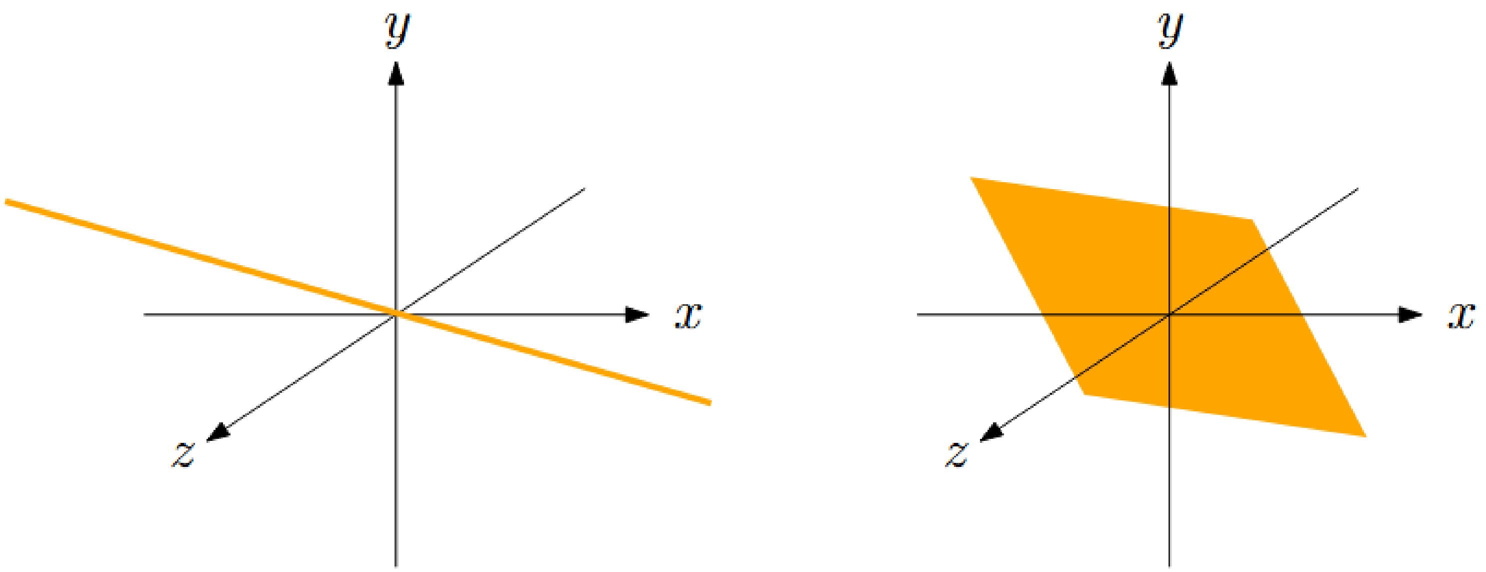
# Unterräume

**Definition 4.8** (Subspace). *Let $V$ be a vector space. A nonempty subset $U \subseteq V$ is called a* subspace *of $V$ if the following two* axioms of a subspace *are true for all $\mathbf{v}, \mathbf{w} \in U$ and all $\lambda \in \mathbb{R}$.*

*(i)* $\mathbf{v} + \mathbf{w} \in U$;

*(ii)* $\lambda \mathbf{v} \in U$.

Unterräume von $\mathbb{R}^3$:



# Unterräume

**Definition 4.8** (Subspace). *Let $V$ be a vector space. A nonempty subset $U \subseteq V$ is called a* subspace *of $V$ if the following two* axioms of a subspace *are true for all $\mathbf{v}, \mathbf{w} \in U$ and all $\lambda \in \mathbb{R}$.*

*(i)* $\mathbf{v} + \mathbf{w} \in U$;

*(ii)* $\lambda \mathbf{v} \in U$.

- Ist die folgende Menge ein Unterraum des angegebenen Vektorraums?

(a) $S_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = 2x_3\} \subseteq \mathbb{R}^3$

✓

# Unterräume

**Definition 4.8** (Subspace). *Let $V$ be a vector space. A nonempty subset $U \subseteq V$ is called a* subspace *of $V$ if the following two* axioms of a subspace *are true for all $\mathbf{v}, \mathbf{w} \in U$ and all $\lambda \in \mathbb{R}$.*

*(i)* $\mathbf{v} + \mathbf{w} \in U$;

*(ii)* $\lambda \mathbf{v} \in U$.

- Ist die folgende Menge ein Unterraum des angegebenen Vektorraums?

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3\} \subseteq \mathbb{R}^3$$

✗

# Unterräume

**Definition 4.8** (Subspace). *Let $V$ be a vector space. A nonempty subset $U \subseteq V$ is called a* subspace *of $V$ if the following two* axioms of a subspace *are true for all $\mathbf{v}, \mathbf{w} \in U$ and all $\lambda \in \mathbb{R}$.*

*(i)* $\mathbf{v} + \mathbf{w} \in U$;

*(ii)* $\lambda \mathbf{v} \in U$.

- Ist die folgende Menge ein Unterraum des angegebenen Vektorraums?

$$\text{(b)} \quad S_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^4 = 0\} \subseteq \mathbb{R}^2$$

✓

# Unterräume

**Definition 4.8** (Subspace). *Let $V$ be a vector space. A nonempty subset $U \subseteq V$ is called a* subspace *of $V$ if the following two* axioms of a subspace *are true for all $\mathbf{v}, \mathbf{w} \in U$ and all $\lambda \in \mathbb{R}$.*

*(i)* $\mathbf{v} + \mathbf{w} \in U$;

*(ii)* $\lambda \mathbf{v} \in U$.

- Ist die folgende Menge ein Unterraum des angegebenen Vektorraums?

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2\} \subseteq \mathbb{R}^2$$

❌

# Unterräume

**Lemma 4.11** (The column space is a subspace). *Let $A$ be an $m \times n$ matrix. Then the column space* $\mathbf{C}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$ *is a subspace of* $\mathbb{R}^m$.

**Corollary 4.12** (The row space is a subspace). *Let $A$ be an $m \times n$ matrix. Then the row space* $\mathbf{R}(A) = \mathbf{C}(A^\top)$ *is a subspace of* $\mathbb{R}^n$.

**Exercise 4.13** (The nullspace is a subspace). *Let $A$ be an $m \times n$ matrix. Then the nullspace* $\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ *is a subspace of* $\mathbb{R}^n$.

# Unterräume sind Vektorräume

**Lemma 4.14** (Subspaces are vector spaces)**.** *Let $V$ be a vector space, and let $U$ be a subspace of $V$. Then $U$ is also a vector space (with the same "$+$" and "$\cdot$" as $V$).*

# Basen und Dimensionen

**Definition 4.18** (Basis). *Let $V$ be a vector space. A subset $B \subseteq V$ is called a* basis *of $V$ if $B$ is linearly independent and* $\mathbf{Span}(B) = V$.

- Problem: Bis jetzt haben wir lineare Unabhängigkeit und Span nur für Vektoren in $\mathbb{R}^n$, nicht für beliebige Vektorräume.

# Was ist linear unabhängig/Span für allgemeine Vektorräume?

**Definition 4.15** (Linear combination of a set of vectors). *Let $V$ be a vector space, $G \subseteq V$ a (possibly infinite) subset of vectors. A* linear combination *of $G$ is a sum of the form*

$$\sum_{j=1}^{n} \lambda_j \mathbf{v}_j,$$

*where $F = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a finite subset of $G$.*

**Definition 4.17** (Linear independence and span of a set of vectors). *Let $V$ be a vector space, $G \subseteq V$ a (possibly infinite) subset of vectors.*

*The set $G$ is called* linearly dependent *if there is an element $\mathbf{v} \in G$ such that $\mathbf{v}$ is a linear combination of $G \setminus \{\mathbf{v}\}$. Otherwise, $G$ is called* linearly independent.

*The* span *of $G$, written as $\mathbf{Span}(G)$, is the set of all linear combinations of $G$.*

# Beispiele für Basen

| vector space $V$ | basis $B$ |
|---|---|
| $\mathbb{R}^m$ | $\{e_1, e_2, \ldots, e_m\}$ |
| $C(A)$ (subspace of $\mathbb{R}^m$) | independent columns of $A$ |
| $2 \times 2$ symmetric matrices (subspace of $R^{2\times 2}$) | $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ |
| $\mathbb{R}[x]$ (polynomials) | $\{x^i : i = 0, 1, \ldots\}$ (infinite set) |
| $\{0\}$ (smallest vector space) | $\emptyset$ (empty set) |

Es gibt oft mehrere Basen für denselben Vektorraum!

# Endlich generierte Vektorräume

**Definition 4.21** (Finitely generated vector space). *A vector space $V$ is called* finitely generated *if there exists a finite subset $G \subseteq V$ with* $\mathbf{Span}(G) = V$.

**Theorem 4.22.** *Let $V$ be a finitely generated vector space, and let $G \subseteq V$ be a finite subset with* $\mathbf{Span}(G) = V$. *Then $V$ has a basis $B \subseteq G$.*

(Auch wenn V nicht endlich generiert ist, hat V eine Basis, aber wir beweisen das nicht in dieser Vorlesung)

# Steinitz Exchange Lemma

**Lemma 4.23** (Steinitz exchange lemma). *Let $V$ be a finitely generated vector space, $F \subseteq V$ a finite set of linearly independent vectors, and $G \subseteq V$ a finite set of vectors with* $\mathbf{Span}(G) = V$. *Then the following two statements hold.*

*(i)* $|F| \leq |G|$.

*(ii) There exists a subset $E \subseteq G$ of size $|G| - |F|$ such that* $\mathbf{Span}(F \cup E) = V$.

**Lemma 1.28** (The span of $m$ linearly independent vectors is $\mathbb{R}^m$). *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^m$ be linearly independent. Then* $\mathbf{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) = \mathbb{R}^m$.

# Dimension

**Theorem 4.24** (All bases have the same size). *Let $V$ be a finitely generated vector space and let $B, B' \subseteq V$ be two bases of $V$. Then $|B| = |B'|$.*

**Definition 4.25** (Dimension). *Let $V$ be a finitely generated vector space. Then $\dim(V)$, the dimension of $V$, is the size of an arbitrary basis $B$ of $V$.*

# Fragen?

# Übungen

## 2. Subspaces of function spaces and $\mathbb{R}^{m\times m}$ (in-class) (★☆☆)

**a)** In this exercise we consider the vector space $V$ of all real-valued functions on the interval $[0,1]$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : [0,1] \to \mathbb{R}$ and conversely, every function $\mathbf{f} : [0,1] \to \mathbb{R}$ is in $V$. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

$$(\mathbf{f}+\mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in [0,1]$$
$$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in [0,1] \text{ and } c \in \mathbb{R}.$$

Prove that

$$U = \{\mathbf{f} \in V : \mathbf{f}(x) = \mathbf{f}(1-x) \text{ for all } x \in [0,1]\} \subseteq V$$

is a subspace of $V$.

**2. Subspaces of function spaces and $\mathbb{R}^{m \times m}$ (in-class) (★☆☆)**

**b)** Let $m \in \mathbb{N}^+$. Consider the set $\mathcal{D}_m$ of diagonal $m \times m$ matrices, which is a subspace of $\mathbb{R}^{m \times m}$. What is the dimension of $\mathcal{D}_m$? Justify your answer with a proof.

# Bemerkungen zu Aufgabe 2

- Vorsicht mit Indizes von Matrizen!

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

*Hence, $a_{ij}$ is the entry in row $i$ and column $j$ of matrix $A$. The "dot-free" notation (see also Section 1.1.5) is*

$$A = [a_{ij}]_{i=1,j=1}^{m\ \ \ n}.$$

# Bemerkungen zu Aufgabe 2

- Repetitions Matrix-Vektor Multiplikation

**Observation 2.8** (Matrix-vector multiplication with $A$ in row notation). *Let*

$$A = \begin{bmatrix} — & \mathbf{u}_1^\top & — \\ — & \mathbf{u}_2^\top & — \\ & \vdots & \\ — & \mathbf{u}_m^\top & — \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad \mathbf{x} \in \mathbb{R}^n. \quad \textit{Then } A\mathbf{x} = \underbrace{\begin{pmatrix} \mathbf{u}_1^\top \mathbf{x} \\ \mathbf{u}_2^\top \mathbf{x} \\ \vdots \\ \mathbf{u}_m^\top \mathbf{x} \end{pmatrix}}_{m \text{ scalar products}} .$$

# Bemerkungen zu Aufgabe 2

- Grundsätzlich verwenden wir in den Aufgaben nur Aussagen, die wir in der Vorlesung oder Übungen gesehen haben.

## 2. Strictly diagonally dominant matrices (hand-in) (★★★)

Let $A \in \mathbb{R}^{m \times m}$ be strictly diagonally dominant, that is,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}|$$

holds for all $i \in [m]$. Prove that $A$ is invertible.

## 6. Linear independence in $\mathbb{R}^3$ (★★☆)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be three linearly independent vectors in $\mathbb{R}^3$. Consider the three vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^3$ defined as

$$\mathbf{w}_1 \coloneqq \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_2 \coloneqq -\mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 \coloneqq \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.$$

Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent.

**4. Invertibility (★☆☆)**

Let $A \in \mathbb{R}^{3\times 3}$ be the following upper triangular matrix with $a, b, c, d \in \mathbb{R}$:

$$A = \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}.$$

For which values of $a, b, c, d$ is $A$ invertible? Specify $A^{-1}$ for these cases.