Bitte setzt euch in den vordersten vier Reihen!

Lineare Algebra

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Programm

- Theorie-Input & In-class Exercise
- Nachbesprechung Serie 4

Hand-In Aufgabe Serie 5

- Sehr schwierig, hat 3 Sterne!
- Schaut euch die Hints an (auf annikaguhl.com)
- Falls ihr die Aufgabe nicht ganz lösen könnt, ist das nicht schlimm, ihr könnt einfach das abgeben, was ihr geschafft habt!

Theorie

Lineare Gleichungssysteme

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_1 - x_3 &= 3 \\ x_1 + x_2 + x_3 &= 17 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 \\ 3 \\ 17 \end{pmatrix}}_{\mathbf{b}} .$$

- Wir können lineare Gleichungssysteme als $A\mathbf{x} = \mathbf{b}$ schreiben

Gauss-Elimination

- Transformiert $Ax = b$ zu $Ux = c$ mit denselben Lösungen, aber U ist eine obere Dreiecksmatrix, bei der die Diagonalelemente nicht 0 sind
- $Ux = c$ kann einfach gelöst werden mit Zurücksubstitution:

Back substitution: if U is upper triangular

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 17 \\ 14 \end{pmatrix} :$$

equation	before substitution	after substitution	solution
1	$2x_1 + 3x_2 + 4x_3 = 19$	$2x_1 + 11 = 19$	$x_1 = 4$
2	$5x_2 + 6x_3 = 17$	$5x_2 + 12 = 17$	$x_2 = 1$
3	$7x_3 = 14$		$x_3 = 2$

Case $m \times m$: Equation $i = m, m-1, \ldots, 1$:

$$\sum_{j=i}^{m} u_{ij} x_j = c_i, \text{ or (when solved for } x_i\text{):} \quad x_i = \frac{c_i - \sum_{j=i+1}^{m} u_{ij} x_j}{u_{ii}}.$$

Gauss-Elimination

- Reihen subtrahieren:

fat number: the **pivot**

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \mathbf{b} = \begin{pmatrix} 19 \\ 55 \\ 50 \end{pmatrix}$$

subtract 2·(row 1) from (row 2):

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} \qquad E_{21}\mathbf{b} = \begin{pmatrix} 19 \\ 17 \\ 50 \end{pmatrix}$$

subtract 1·(row 1) from (row 3):

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{5} & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \begin{pmatrix} 19 \\ 17 \\ 31 \end{pmatrix}$$

subtract 1·(row 2) from (row 3):

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}A}_{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & \mathbf{7} \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}\mathbf{b}}_{\mathbf{c}} = \begin{pmatrix} 19 \\ 17 \\ 14 \end{pmatrix}$$

↑ elimination matrices

done!

Gauss-Elimination

- Was wenn Pivot = 0?
- Reihen vertauschen:

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \mathbf{b} = \cdots$$

elimination in column 1: $\downarrow$ $\qquad \downarrow$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{0} & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \cdots$$

pivot is **0**: exchange (row 2) and (row 3): $\downarrow$ $\qquad \downarrow$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underbrace{P_{23}E_{31}E_{21}A}_{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{5} & 13 \\ 0 & 0 & 6 \end{bmatrix} \qquad \underbrace{P_{23}E_{31}E_{21}\mathbf{b}}_{\mathbf{c}} = \cdots$$

$\uparrow$ permutation matrix

done!

Gauss-Elimination

- Funktioniert nicht immer:

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ \color{red}{4} & 6 & 14 \\ \color{red}{2} & \color{red}{3} & 17 \end{bmatrix} \qquad \mathbf{b} = \cdots$$

elimination in column 1: $\downarrow \qquad \downarrow$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ \color{blue}{0} & \color{red}{\mathbf{0}} & 6 \\ \color{blue}{0} & \color{red}{0} & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \cdots$$

pivot is $\color{red}{\mathbf{0}}$, no row exchange helps: elimination fails.

$$\begin{bmatrix} 2 & 3 & 4 \\ \color{blue}{0} & 5 & 6 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{\mathbf{0}} \end{bmatrix}$$

we can also fail in the last column!

Gauss-Elimination Beispiel

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 4 \\ 2 & 2 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 15 \end{pmatrix}$$

1. Interpolation (in-class) (★☆☆) Assume that you have collected the following data points

x	y
0	1
2	2
4	5
6	6

and you want to find a function $f : \mathbb{R} \to \mathbb{R}$ that interpolates them, i.e. f should satisfy $f(x) = y$ for all pairs of x, y given by the table above. There is an abundance of functions that you can try and in particular, there are many different functions that do interpolate the four datapoints. In this exercise, we are interested in polynomials, i.e. we restrict f to be a polynomial of degree at most 3. In particular, this means that f has the form $f(x) = ax^3 + bx^2 + cx + d$ for some $a, b, c, d \in \mathbb{R}$. Your task is to find values for a, b, c, d such that f interpolates all four points given in the table.

Eigenschaften von Reihenoperationen

Lemma 3.2 (Invariance of solutions): Let A be an $m \times n$ matrix, M an invertible $m \times m$ matrix, $\mathrm{b} \in \mathbb{R}^m$. The two systems $A\mathrm{x} = \mathrm{b}$ and $MA\mathrm{x} = M\mathrm{b}$ have the same solutions $\mathbf{x}$.

Lemma 3.3 (Invariance of the nullspace): A, M as before. Then A and MA have the same nullspace, $\mathbf{N}(A) = \mathbf{N}(MA)$.

Lemma 3.4 (Invariance of linear independence): A, M as before. A has linearly independent columns if and only if MA has linearly independent columns.

Eigenschaften von Reihenoperationen

Lemma 3.5 (Invariance of the row space). *Let A be an $m \times n$ matrix and M an invertible $m \times m$ matrix. Then A and MA have the same row space,* $\mathbf{R}(A) = \mathbf{R}(MA)$.

Lemma 3.6 (Invariance of independent column indices and rank): A, M as before. For all $j \in [n]$: column j of A is independent $\Leftrightarrow$ column j of MA is independent. So A and MA have the same number of independent columns and therefore the same rank.

Lemma 3.5 (Invariance of the row space)**.** *Let A be an $m \times n$ matrix and M an invertible $m \times m$ matrix. Then A and MA have the same row space, $\mathbf{R}(A) = \mathbf{R}(MA)$.*

Erfolg der Gauss-Elimination

Theorem 3.7. *Let* $A\mathbf{x} = \mathbf{b}$ *be a system of* m *linear equations in* m *variables (so* A *is an* $m \times m$ *matrix). The following two statements are equivalent.*

(i) Gauss elimination as in Algorithm 2 *succeeds.*

(ii) The columns of A *are linearly independent.*

Gauss-Elimination Beispiel 2

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ 1 & 3 & -2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 17 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ 1 & 3 & -2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 23 \\ 12 \end{pmatrix}$$

Gauss-Elimination für m rechte Seiten

Algorithm 3 Gauss elimination with m right-hand sides:
Returns a triple (U, C, result) such that for all $j \in [m]$, the two systems $A\mathbf{x} = \mathbf{b}_j$ and $U\mathbf{x} = \mathbf{c}_j$ have the same solutions, where $\mathbf{b}_j$ is the j-th column of B, and $\mathbf{c}_j$ the j-th column of C. If result = "succeeded", U is upper triangular with all diagonal elements nonzero.

function GAUSS ELIMINATION(A, B) ▷ $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{m\times m}$
    $U \leftarrow A, C \leftarrow B$
    for $j = 1, 2, \ldots, m$ **do** ▷ eliminate in column j
        if $u_{jj} = 0$ **then** ▷ zero pivot
            if there is some $k > j$ such that $u_{kj} \neq 0$ **then**
                exchange (row j) and (row k) (in both U and C) ▷ row operation
            else ▷ give up
                return $(U, C, \text{"failed"})$
            end if
        end if ▷ now, $u_{jj} \neq 0$
        for $i = j+1, j+2, \ldots, m$ **do** ▷ make $u_{ij} = 0$
            $\lambda \leftarrow \frac{u_{ij}}{u_{jj}}$ ▷ we want $u_{ij} - \lambda u_{jj} = 0$
            subtract $\lambda \cdot$ (row j) from (row i) (in both U and C) ▷ row operation
        end for
    end for ▷ now, U is upper triangular, with all diagonal elements nonzero
    return $(U, C, \text{"succeeded"})$
end function

Gauss-Elimination für m rechte Seiten

- Genau das gleiche wie Gauss-Elimination mit einer rechten Seite, nur dass man gleichzeitig mehrere rechte Seiten berechnet

Algorithm 3 Gauss elimination with m right-hand sides:
Returns a triple (U, C, result) such that for all $j \in [m]$, the two systems $A\mathbf{x} = \mathbf{b}_j$ and $U\mathbf{x} = \mathbf{c}_j$ have the same solutions, where $\mathbf{b}_j$ is the j-th column of B, and $\mathbf{c}_j$ the j-th column of C. If result = "succeeded", U is upper triangular with all diagonal elements nonzero.

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    if $u_{jj} = 0$ **then** ▷ zero pivot
      if there is some $k > j$ such that $u_{kj} \neq 0$ **then**
        exchange (row j) and (row k) (in both U and C) ▷ row operation
      else ▷ give up
        return $(U, C, \text{"failed"})$
      end if
    end if ▷ now, $u_{jj} \neq 0$
    for $i = j+1, j+2, \ldots, m$ **do** ▷ make $u_{ij} = 0$
      $\lambda \leftarrow \frac{u_{ij}}{u_{jj}}$ ▷ we want $u_{ij} - \lambda u_{jj} = 0$
      subtract $\lambda \cdot$ (row j) from (row i) (in both U and C) ▷ row operation
    end for
  end for ▷ now, U is upper triangular, with all diagonal elements nonzero
  return $(U, C, \text{"succeeded"})$
end function

Beispiel Gauss-Elimination mit m rechten Seiten: Inverse berechnen

Mache Gauss-Elimination mit $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ -1 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix-Inverse berechnen

- Gauss-Elimination für m rechte Seiten mit $B = I$

Algorithm 4 Inverse matrix computation:
returns a pair (M, result) such that result = "succeeded" if and only if A is invertible. If result = "succeeded", then $M = A^{-1}$.

function INVERSE(A)
  $(U, C, \text{result}) \leftarrow$ GAUSS ELIMINATION(A, I) ▷ $A, I \in \mathbb{R}^{m \times m}$
  if result = "failed" **then**
    return (A, "singular")
  else
    for $j = 1, 2, \ldots, m$ **do** ▷ solve $U\mathbf{x} = \mathbf{c}_j$ which is equivalent to $A\mathbf{x} = \mathbf{e}_j$
      $\mathbf{c}_j \leftarrow j$-th column of C
      $\mathbf{x}_j \leftarrow$ BACK SUBSTITUTION($U, \mathbf{c}_j$)
    end for
    $$A^{-1} \leftarrow \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_m \\ | & | & & | \end{bmatrix}$$
    return (A^{-1}, "succeeded")
  end if
end function

Fragen?

Übungen

3. Matrix multiplication and invertibility (hand-in) (★★☆)

Let $A, B, C \in \mathbb{R}^{m \times m}$ such that $BA = CA$.

a) Suppose that A is invertible. Show that $B = C$.

b) Is it true that $AB = AC$? Give either a proof or a counterexample.

c) Suppose that $B - C$ is invertible. Show that $A = 0$.

5. Inverses of matrix powers (★★☆)

a) Let A be an $m \times m$ matrix with inverse A^{-1} and let $k \in \mathbb{N}^+$ be an arbitrary integer. Does A^k have an inverse and if yes, what is it?

b) Recall the definition of a nilpotent matrix: We say that a square matrix A is nilpotent if and only if there exists $k \in \mathbb{N}$ such that $A^k = 0$. Prove that a nilpotent matrix A cannot have an inverse.

c) Let A be an $m \times m$ matrix with $A^3 = I$ and $A^4 = I$. Prove that $A = I$.

d) Find a 2×2 matrix $A \neq I$ such that $A^k = I$ for all even k and $A^k = A$ for all odd $k \in \mathbb{N}$.

e) Can you also find a 2×2 matrix A that, for all $k \in \mathbb{N}$, satisfies $A^k = I$ if and only if $k \equiv_4 0$ (i.e. $A^k = I$ if and only if k is a multiple of 4)?

6. Inverse of triangular matrices (★★★)

a) Find the inverse of the 2×2 matrix $L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ where $a \in \mathbb{R}$.

b) Prove that a square lower triangular matrix is invertible if and only if all its diagonal entries are non-zero.