Bitte setzt euch in den vordersten vier Reihen!

Lineare Algebra

Übung 4, 16. Oktober 2025

Annika Guhl, annikaguhl.com

Programm

- Theorie-Input
- In-class Exercise
- Nachbesprechung Serie 3

Theorie

Alles ist eine Matrixmultiplikation

(Mixed) products (e.g. $\mathbf{x}^\top A^\top A\mathbf{x}$) can be evaluated (using matrix multiplication) as if

- vectors in $\mathbb{R}^m$ were $m \times 1$ matrices,
- covectors in $(\mathbb{R}^n)^*$ were $1 \times n$ matrices,
- scalars in $\mathbb{R}$ were 1×1 matrices.

Definition 2.44. *Let* $\mathbf{v} \in \mathbb{R}^m, \mathbf{w} \in \mathbb{R}^n$. *The* outer product $\mathbf{v}\mathbf{w}^\top$ *of* $\mathbf{v}$ *and* $\mathbf{w}$ *is the* $m \times n$ *matrix*

$$\mathbf{v}\mathbf{w}^\top := \begin{bmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{bmatrix} = [v_i w_j]_{i=1,j=1}^{m\ \ \ n}.$$

Lemma 2.15 (Rank-1 matrices). *Let* A *be an* $m \times n$ *matrix. The following two statements are equivalent.*

(i) $\mathbf{rank}(A) = 1$.

(ii) *There are nonzero vectors* $\mathbf{v} \in \mathbb{R}^m, \mathbf{w} \in \mathbb{R}^n$ *such that*

$$A = [v_i w_j]_{i=1,j=1}^{m\ \ \ n}.$$

CR-Dekomposition

Theorem 2.46 (CR decomposition). *Let A be an $m \times n$ matrix of rank r (Definition 2.10). Let C be the $m \times r$ submatrix of A containing the independent columns. Then there is a unique $r \times n$ matrix R' such that*

$$A = CR'.$$

CR-Dekomposition

Theorem 2.46 (CR decomposition)**.** *Let A be an $m \times n$ matrix of rank r (Definition 2.10). Let C be the $m \times r$ submatrix of A containing the independent columns. Then there is a unique $r \times n$ matrix R' such that*

$$A = CR'.$$

- Beispiel: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

CR-Dekomposition

Theorem 2.46 (CR decomposition). *Let A be an $m \times n$ matrix of rank r (Definition 2.10). Let C be the $m \times r$ submatrix of A containing the independent columns. Then there is a unique $r \times n$ matrix R' such that*

$$A = CR'.$$

- Beispiel: $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$

Injektiv, surjektiv, bijektiv

Definition 2.48 (Injective, surjective, and bijective functions). *Let X, Y be sets and $f : X \to Y$ a function.*

(i) *f is called* injective *if for every $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.* *("For every possible output, at most one input leads to it.")*

(ii) *f is called* surjective *if for every $y \in Y$, there is at least one $x \in X$ with $f(x) = y$.* *("For every possible output, at least one input leads to it.")*

(iii) *f is called* bijective *(undoable) if f is both injective and surjective.* *("For every possible output, exactly one input leads to it.")*

Lernt diese Definitionen auswendig!

Injektiv, surjektiv, bijektiv

Definition 2.48 (Injective, surjective, and bijective functions). *Let X, Y be sets and $f : X \to Y$ a function.*

(i) f is called injective *if for every $y \in Y$, there is at most one $x \in X$ with $f(x) = y$. ("For every possible output, at most one input leads to it.")*

(ii) f is called surjective *if for every $y \in Y$, there is at least one $x \in X$ with $f(x) = y$. ("For every possible output, at least one input leads to it.")*

(iii) f is called bijective *(undoable) if f is both injective and surjective. ("For every possible output, exactly one input leads to it.")*

Ist diese Funktion injektiv, surjektiv, bijektiv oder nichts davon?

$$f\colon \mathbb{R} \to \mathbb{R}, f(x) = 2x$$

Bijektiv!

Injektiv, surjektiv, bijektiv

Definition 2.48 (Injective, surjective, and bijective functions). *Let X, Y be sets and $f : X \to Y$ a function.*

(i) *f is called* injective *if for every $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.*
("For every possible output, at most one input leads to it.")

(ii) *f is called* surjective *if for every $y \in Y$, there is at least one $x \in X$ with $f(x) = y$.*
("For every possible output, at least one input leads to it.")

(iii) *f is called* bijective *(undoable) if f is both injective and surjective.*
("For every possible output, exactly one input leads to it.")

Ist diese Funktion injektiv, surjektiv, bijektiv oder nichts davon?

$$f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2$$

Weder noch!

Injektiv, surjektiv, bijektiv

Definition 2.48 (Injective, surjective, and bijective functions). *Let X, Y be sets and $f : X \to Y$ a function.*

(i) *f is called* injective *if for every $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.*
("For every possible output, at most one input leads to it.")

(ii) *f is called* surjective *if for every $y \in Y$, there is at least one $x \in X$ with $f(x) = y$.*
("For every possible output, at least one input leads to it.")

(iii) *f is called* bijective *(undoable) if f is both injective and surjective.*
("For every possible output, exactly one input leads to it.")

Ist diese Funktion injektiv, surjektiv, bijektiv oder nichts davon?

$$f\colon \mathbb{R} \to [0, \infty), f(x) = x^2$$

Surjektiv, nicht injektiv

Injektiv, surjektiv, bijektiv

Definition 2.48 (Injective, surjective, and bijective functions). *Let X, Y be sets and $f : X \to Y$ a function.*

(i) *f is called* injective *if for every $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.*
("For every possible output, at most one input leads to it.")

(ii) *f is called* surjective *if for every $y \in Y$, there is at least one $x \in X$ with $f(x) = y$.*
("For every possible output, at least one input leads to it.")

(iii) *f is called* bijective *(undoable) if f is both injective and surjective.*
("For every possible output, exactly one input leads to it.")

Ist diese Funktion injektiv, surjektiv, bijektiv oder nichts davon?

$$f\colon \mathbb{R} \to [0, \infty), f(x) = \mathrm{e}^x$$

Injektiv, nicht surjektiv

Inverse einer Funktion

(iv) The inverse *of a bijective function f is the function*

$$f^{-1} : Y \to X, \quad y \mapsto \textit{the unique } x \in X \textit{ such that } f(x) = y.$$

Beispiele von Inversen von Funktionen

(iv) The inverse *of a bijective function f is the function*

$$f^{-1} : Y \to X, \quad y \mapsto \textit{the unique } x \in X \textit{ such that } f(x) = y.$$

$f(x)$	$f(x)^{-1}$
$x + a$	$x - a$
$ax, a \neq 0$	$\frac{x}{a}, a \neq 0$
e^x	$\ln x$
$\sin x$	$\sin^{-1} x$

Inverse einer Bijektion

Fact 2.49 (Bijective functions and their inverses). *If $f : X \to Y$ is bijective, then $f^{-1} : Y \to X$ is also bijective, and $(f^{-1})^{-1} = f$. Moreover, $f^{-1} \circ f = \mathrm{id}$ (f^{-1} is undoing f) and $f \circ f^{-1} = \mathrm{id}$ (f is undoing f^{-1}).*

Lemma 2.52 (The inverse of a bijective linear transformation). *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a bijective linear transformation. Then its inverse $T^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is also a linear transformation (and bijective by Fact 2.49).*

Inverse einer Matrix

Definition 2.57 (Inverse matrix). *Let A be an $m \times m$ matrix. A is invertible if and only if there exists an $m \times m$ matrix B such that $BA = I$ (or $AB = I$, or $AB = BA = I$). In this case, the matrix B is unique and called the* inverse *of A. We denote it by A^{-1}.*

Inverse einer Matrix

Case 1×1**.**

$$A = \begin{bmatrix} a \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} \frac{1}{a} \end{bmatrix} \quad (\text{if } a \neq 0).$$

Case 2×2**.**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{if } ad - bc \neq 0).$$

Case $m \times m$**.** Inefficient formula (based on *Cramer's rule*); efficient computation in Chapter 3.

Inverse einer Matrix

Case 1×1**.**

$$A = [a] \quad \Rightarrow \quad A^{-1} = \left[\tfrac{1}{a}\right] \quad (\text{if } a \neq 0).$$

Case 2×2**.**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{if } ad - bc \neq 0).$$

- Was ist die Inverse von $A = [4]$?
- $A^{-1} = [0.25]$

Inverse einer Matrix

Case 1×1**.**

$$A = [a] \quad \Rightarrow \quad A^{-1} = \left[\tfrac{1}{a}\right] \quad \text{(if } a \neq 0\text{)}.$$

Case 2×2**.**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(if } ad - bc \neq 0\text{)}.$$

- Was ist die Inverse von $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$?
- $A^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

Inverse einer Matrix

Lemma 2.59. *Let A and B be invertible $m \times m$ matrices. Then AB is also invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Lemma 2.60. *Let A be an invertible $m \times m$ matrix. Then the transpose $A^\top$ is also invertible, and*

$$\left(A^\top\right)^{-1} = \left(A^{-1}\right)^\top.$$

Inverse einer Matrix

Lemma 2.60. *Let A be an invertible $m \times m$ matrix. Then the transpose $A^\top$ is also invertible, and*

$$\left(A^\top\right)^{-1} = \left(A^{-1}\right)^\top .$$

Fragen?

Übungen

1. Matrix multiplication with vectors and covectors (in-class) (★☆☆)

Let $n \in \mathbb{N}$. Consider $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ given by $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$.

a) Compute $\mathbf{v}^\top \mathbf{w}$ for all $n \in \mathbb{N}$.

b) Compute $\mathbf{v}\mathbf{w}^\top$ when $n = 4$.

c) Compute $\mathbf{w}^\top(\mathbf{v}\mathbf{w}^\top)\mathbf{v}$ for all $n \in \mathbb{N}$.

2. Exercise 2.47 (in-class) (★☆☆)

Let $A \in \mathbb{R}^{m \times n}$ of rank r and $C \in \mathbb{R}^{m \times r}$ and $R' \in \mathbb{R}^{r \times n}$ be the matrices in the CR-decomposition of A as given in Theorem 2.46.

a) Suppose $r = n$. What are the matrices C and R'?

b) Suppose $r = 0$. What are the matrices C and R'?

2. Matrix powers (bonus, hand-in) (★☆☆)

For a natural number $k \geq 1$, we define the k-th power of a square matrix A as the matrix multiplication

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Moreover we define $A^0 = I$, where I is the identity matrix.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Use induction to show that for any $k \geq 0$ the k-th power of the matrix A is

$$A^k = \begin{bmatrix} 1+k & k \\ -k & 1-k \end{bmatrix}$$

2. Matrix powers (bonus, hand-in) (★☆☆)

For a natural number $k \geq 1$, we define the k-th power of a square matrix A as the matrix multiplication

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Moreover we define $A^0 = I$, where I is the identity matrix.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Use induction to show that for any $k \geq 0$ the k-th power of the matrix A is

$$A^k = \begin{bmatrix} 1+k & k \\ -k & 1-k \end{bmatrix}$$

3. Reconstruct a linear transformation (★☆☆)

a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Determine the general formula for $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$ with $x, y \in \mathbb{R}$.

b) Find a matrix A such that $T_A = T$.

5. Matrix multiplication (★★★)

a) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find $x, y, z \in \mathbb{R}$ such that $A^3 + xA^2 + yA + zI = 0$. Note that both I and 0 are 3×3 matrices in this equation.

b) Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

We say that a square matrix A is *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the *nilpotent degree* of A.

c) Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A. In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB?

d) Let A be an $m \times m$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I - A)(I + A + \ldots + A^{k-1}) = I$.

e) Let T be an $m \times m$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^m = 0$, i.e. T is nilpotent of degree less or equal to m.

4. Linear transformation (★☆☆)

Let $m, n \in \mathbb{N}^+$ and consider an arbitrary $m \times (n+1)$ matrix

$$A = \begin{bmatrix} | & | & & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{v}_{n+1} \\ | & | & & | & | \end{bmatrix}$$

with columns $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{R}^m$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the function defined by

$$T : \mathbf{x} \mapsto A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

for all $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top \in \mathbb{R}^n$. Prove that T is a linear transformation if and only if $\mathbf{v}_{n+1} = \mathbf{0}$.