Bitte setzt euch in den vordersten drei Reihen!

Lineare Algebra

Übung 2, 2. Oktober 2025

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Organisatorisches

- Webseite neu auf annikaguhl.com
- n.ethz.ch/~anguhl ab 1. November nur noch im ETH-Netz erreichbar, ab 2026 offline!
- Neu: Hints zu den Übungsaufgaben auf der Webseite
- Erinnerung: Office Hour am Montag 11:00-13:00 im HG G 26.3, dort könnt ihr Fragen zur Übungsserie stellen!

Programm

- Theorie-Input
- In-class Exercise
- Nachbesprechung Serie 1

Mehr zu Beweisen

- Es braucht Übung!
- https://ti.inf.ethz.ch/ew/courses/LA24/mathe-vorschau/language-and-logic.html
- https://www.logik.uni-jena.de/logikmedia/75/einfuehrung-in-das-mathematische-beweisen.pdf?nonactive=1&suffix=pdf

Fragen?

Theorie

Matrizen

Definition 2.1 (Matrix). *An $m \times n$ matrix is a table of real numbers with m rows and n columns. We use upper-case letters $(A, B, \ldots)$ to denote matrices, and write their entries with the corresponding lower case letters and two indices, as in*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Hence, a_{ij} is the entry in row i and column j of matrix A. The "dot-free" notation (see also Section 1.1.5) is

$$A = [a_{ij}]_{i=1,j=1}^{m \quad n}.$$

The set of $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$.

Matrizen

If we want to talk about the columns of A as vectors or the rows of A as covectors, we use column notation *or* row notation,

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}}_{\substack{\text{column notation with} \\ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m}}, \qquad A = \underbrace{\begin{bmatrix} — & \mathbf{u}_1^\top & — \\ — & \mathbf{u}_2^\top & — \\ & \vdots & \\ — & \mathbf{u}_m^\top & — \end{bmatrix}}_{\substack{\text{row notation with} \\ \mathbf{u}_1^\top, \mathbf{u}_2^\top, \ldots, \mathbf{u}_m^\top \in (\mathbb{R}^n)^*}} \quad .$$

Here,

$$\mathbf{v}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \text{ (}j\text{-th column), and} \quad \mathbf{u}_i^\top = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \text{ (}i\text{-th row).}$$

Operationen mit Matrizen

Matrix addition, matrix scalar multiplication: Definition 2.2

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}.$$

0: zero matrix (all entries are 0)

Besondere Matrizen

Square matrices: Definition 2.3

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 7 & 5 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 7 \\ 0 & 7 & 5 \end{bmatrix}$
identity I	diagonal	upper triangular	lower triangular	symmetric
$a_{ij} = \delta_{ij}$	$i \neq j : a_{ij} = 0$	$i > j : a_{ij} = 0$	$i < j : a_{ij} = 0$	$a_{ij} = a_{ji}$

Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Matrix-Vektor Multiplikation

Definition 2.4 (Matrix-vector multiplication with A in column notation). *Let*

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

The vector

$$A\mathbf{x} := \sum_{j=1}^{n} x_j \mathbf{v}_j \in \mathbb{R}^m$$

is the product *of A and* $\mathbf{x}$.

$$\underbrace{7\begin{pmatrix}1\\3\\5\end{pmatrix} + 8\begin{pmatrix}2\\4\\6\end{pmatrix}}_{\text{linear combination}} = \underbrace{\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{pmatrix}7\\8\end{pmatrix}}_{\text{matrix-vector product}} = \underbrace{\begin{pmatrix}23\\53\\83\end{pmatrix}}_{\text{result}}.$$

Spaltenraum

Definition 2.9 (Column space)**.** *Let A be an $m \times n$ matrix. The* column space $\mathbf{C}(A)$ *of A is the span (set of all linear combinations) of the columns,*

$$\mathbf{C}(A) := \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Spaltenrang

Definition 2.10: Let

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

Column $\mathbf{v}_j$ is *independent* if $\mathbf{v}_j$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$. Otherwise, $\mathbf{v}_j$ is *dependent*. The *rank* of A, $\mathbf{rank}(A)$, is the number of independent columns.

$$\mathbf{rank}\left(\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\right) =$$

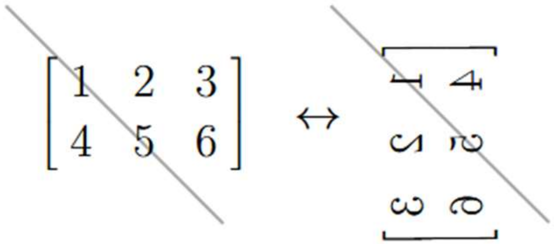
Die Transponierte

Definition 2.12 (Transpose). *Let* $A = [a_{ij}]_{i=1,j=1}^{m\ \ n}$ *be an* $m \times n$ *matrix. The* transpose *of* A *is the* $n \times m$ *matrix*

$$A^\top := B = [b_{ij}]_{i=1,j=1}^{n\ \ m},$$

where $b_{ij} = a_{ji}$ *for all* i, j.

Transposition: Mirroring a matrix along the diagonal:



$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \leftrightarrow \quad A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Reihenraum

Definition 2.14: Let A be an $m \times n$ matrix. The *row space* $\mathbf{R}(A)$ of A is the column space of the transpose,

$$\mathbf{R}(A) := \mathbf{C}(A^\top) \subseteq \mathbb{R}^n.$$

Nullraum

Definition 2.17 (Nullspace). *Let A be an $m \times n$ matrix. The* nullspace *of A is the set*

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

$\mathbf{0} \in \mathbf{N}(A)$ $(A\mathbf{0} = \mathbf{0})$.
$\mathbf{N}(A) = \{\mathbf{0}\} \Leftrightarrow$ columns of A are linearly independent
$\mathbf{N}(A) = \mathbb{R}^n \Leftrightarrow A = 0$.

Matrix-Transformation

Definition 2.18: Let A be an $m \times n$ matrix. The function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A : \underbrace{\mathbf{x}}_{\text{"input"} \in \mathbb{R}^n} \mapsto \underbrace{A\mathbf{x}}_{\text{"output"} \in \mathbb{R}^m}$$

is the *matrix transformation* with matrix A.

Lemma 2.19 (Linearity of matrix transformations). *Let A be an $m \times n$ matrix, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then*

$$A(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) = \lambda_1 A\mathbf{x}_1 + \lambda_2 A\mathbf{x}_2.$$

Fragen?

Übungen

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \ldots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

b) Determine the rank of A. You need to motivate your answer.

2. Orthogonality and Linear independence (hand-in) (★☆☆)

a) For which number $s \in \mathbb{R}$ are the two following vectors orthogonal?

$$\mathbf{v} = \begin{pmatrix} s \\ 3 \\ 2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ s \end{pmatrix}$$

b) For which number $t \in \mathbb{R}$ are the three following vectors linearly dependent?

$$\mathbf{u} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

c) Show that if two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal, then they are linearly independent. Prove that the converse is not necessarily true, i.e. provide an example of two linearly independent vectors that are not orthogonal.

4. Linear independence (★★☆)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m \in \mathbb{R}^m$ be the standard unit vectors in $\mathbb{R}^m$. Consider the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^m$ with $\mathbf{v}_i := \mathbf{e}_i + \mathbf{e}_{i+1}$ for all $i \in \{1, 2, \ldots, m-1\}$ and $\mathbf{v}_m := \mathbf{e}_m + \mathbf{e}_1$.

For example, we get

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

in the case $m = 3$, and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

in the case $m = 4$.

a) Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent if m is even.

b) Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent if m is odd.

5. Angle between two vectors (★★★)

Consider two non-zero vectors $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$ in $\mathbb{R}^3$ with $x + y + z = 0$. Determine the value of $\cos(\alpha)$ where α denotes the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. You are not required to compute (or look up) α, but you are of course welcome to do so.