1. Properties of pseudoinverses (in-class) (★★☆)

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ be arbitrary matrices.

a) Prove that if $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, we have $(AB)^\dagger = B^\dagger A^\dagger$.

b) Prove that $A^\dagger A A^\dagger = A^\dagger$.

c) Prove that $(A^\top)^\dagger = (A^\dagger)^\top$.

Proposition 6.4.9. *For $A \in \mathbb{R}^{m\times n}$, with $\operatorname{rank}(A) = r$, let $S \in \mathbb{R}^{m\times r}$ and $T \in \mathbb{R}^{r\times n}$ such that $A = ST$.*

$$A^\dagger = T^\dagger S^\dagger.$$

a) Wir verwenden Proposition 6.4.9:

Sei $M = AB$, $M \in \mathbb{R}^{m\times p}$

Wir müssen also zeigen, dass $\operatorname{rank}(M) = n$, dann können wir 6.4.9 anwenden. Da $B \in \mathbb{R}^{n\times p}$ und $\operatorname{rank}(B) = n$ folgt, dass B linear unabhängige Reihen hat. Da $AB = [A b_1 \dots A b_n]$, $B = [b_1 \dots b_2]$ und $C(B) = \mathbb{R}^n$, gilt $C(M) = C(AB) = \{(AB)x \mid x \in \mathbb{R}^p\} =$

$$\{A(Bx) \mid x \in \mathbb{R}^p\} = \{Ay \mid y \in C(B)\} = \{Ay \mid y \in \mathbb{R}^n\} = C(A).$$

Also gilt $C(M) = C(A)$, woraus folgt, dass $\operatorname{rank}(A) = n$.

Wir wenden nun Prop. 6.4.9 und erhalten, dass

$$(AB)^+ = M^+ = B^+ A^+.$$

b) Es gilt:

$$A^+ A A^+ = (R^+ C^+)(C R)(R^+ C^+) \qquad \text{6.4.7 (Def. Pseudoinverse)}$$

$$= R^+(C^+ C)(R R^+) C^+ \qquad \text{(Assoziativität Matrixmultiplikation)}$$

$$= R^+ I (R R^+) C^+$$

$$= R^+ I I C^+$$

$$= R^+ C^+$$

$$= A^+$$

Proposition 6.4.2. *For $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = n$, the pseudoinverse $A^\dagger$ is a left inverse of A, meaning that $A^\dagger A = I$.*

Lemma 6.4.4. *For $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = m$, the pseudoinverse $A^\dagger$ is a right inverse of A, meaning that $AA^\dagger = I$.*

c) $(A^T)^+ = ((CR)^T)^+ \overset{\text{Lemma 2.40}}{=} (R^T C^T)^+ \overset{\text{Prop. 6.4.9}}{=} (C^T)^+ (R^T)^+$

$(A^+)^T = (R^+ C^+)^T = (C^+)^T (R^+)^T$

Es gilt:

$C^+ = (C^T C)^{-1} C^T$ (C hat vollen Spaltenrang)

$\Rightarrow (C^+)^T = ((C^T C)^{-1} C^T)^T$

$= C ((C^T C)^{-1})^T$

$= C ((C^T C)^T)^{-1}$ Lemma 2.60: $(A^T)^{-1} = (A^{-1})^T$

$= C (C^T C)^{-1}$

$= (C^T)^+$

Ebenso für R:

$R^+ = R^T (R R^T)^{-1}$

$\Rightarrow (R^+)^T = (R^T (R R^T)^{-1})^T$

$= ((R R^T)^{-1})^T R$

$= ((R R^T)^T)^{-1} R$

$= (R R^T)^{-1} R$

$= (R^T)^+$

Also gilt $(A^T)^+ = (C^T)^+ (R^T)^+ = (C^+)^T (R^+)^T = (A^+)^T$