1. Linear regression (in-class) (★☆☆)

We want to determine the parameters of a certain model function from some measured values. Assume that we measured the following values

t_i	1	2	3	4	5
b_i	2	3	5	6	8

where $i \in [5]$. Moreover, assume that we want to model the relationship between t, b by a function f, i.e. $b = f(t)$. In this exercise, we restrict f to be a line, i.e. f should have the form

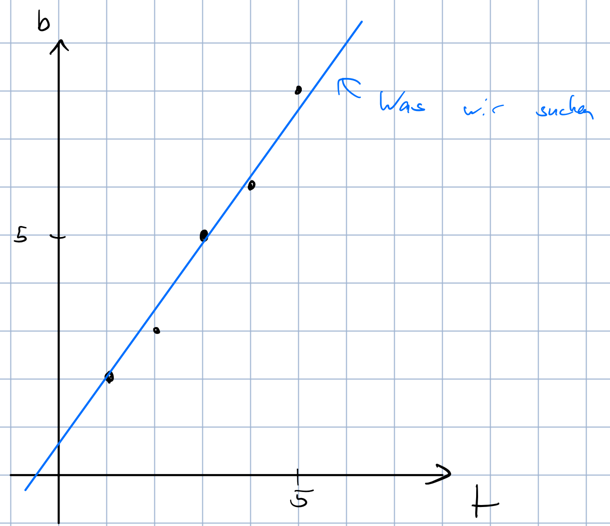
$$f(t) = \alpha_1 t + \alpha_0$$

for parameters $\alpha_1, \alpha_0 \in \mathbb{R}$. Our goal is to find suitable values for α_1, α_0. As discussed in the lecture, this idea of fitting a line through a set of datapoints is called linear regression.

a) For each datapoint (t_i, b_i) with $i \in [5]$, we get an equation for α_1, α_0 from $f(t_i) = b_i$. Write down the system of linear equations that we get by combining all five equations.

b) Do you expect this system to have any solutions? (Answer this intuitively without actually solving the system).

c) Using the normal equations, find an approximate solution to the system you wrote down.



a)

$$\alpha_1 \cdot 1 + \alpha_0 = 2$$

$$\alpha_1 \cdot 2 + \alpha_0 = 3$$

$$\vdots$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \\ 8 \end{pmatrix}$$

b) Nein, wir haben 5 Gleichungen aber nur zwei Unbekannte.

Wir erwarten also nicht, dass das System eine Lösung hat.

c) Eine Approximation finden wir, indem wir $A^T A x = A^T b$ lösen.

$$A^T A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 87 \\ 24 \end{pmatrix}$$

Also erhalten wir das folgende Gleichungssystem:

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 87 \\ 24 \end{pmatrix}$$

Das kann man z.B. mit Gauss-Elimination machen.

Wir ziehen 3-mal die zweite Reihe von der ersten ab und erhalten:

$$\begin{bmatrix} 10 & 0 \\ 15 & 5 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 24 \end{pmatrix}$$

Daraus sehen wir $\alpha_1 = \frac{3}{2}$ und

$$15 \cdot \frac{3}{2} + 5\alpha_2 = 24 \Rightarrow \alpha_2 = \frac{3}{10}.$$

Also ist $\frac{3}{2}x + \frac{3}{10}$ die beste Approximation in der Form einer Linie.