1. Interpolation (in-class) (★☆☆) Assume that you have collected the following data points

x	y
0	1
2	2
4	5
6	6

and you want to find a function $f : \mathbb{R} \to \mathbb{R}$ that interpolates them, i.e. f should satisfy $f(x) = y$ for all pairs of x, y given by the table above. There is an abundance of functions that you can try and in particular, there are many different functions that do interpolate the four datapoints. In this exercise, we are interested in polynomials, i.e. we restrict f to be a polynomial of degree at most 3. In particular, this means that f has the form $f(x) = ax^3 + bx^2 + cx + d$ for some $a, b, c, d \in \mathbb{R}$. Your task is to find values for a, b, c, d such that f interpolates all four points given in the table.

$$a \cdot 0 + b \cdot 0 + c \cdot 0 + d = 1$$

$$a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 2$$

$$a \cdot 4^3 + b \cdot 4^2 + c \cdot 4 + d = 5$$

$$a\, 6^3 + b\, 6^2 + c\, 6 + d = 6$$

In Matrix-Form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 6 & 36 & 216 \end{bmatrix} \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \end{pmatrix} \qquad (*)$$

Lösen mit Gauss-Elimination:

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 4 & 8 & 2 \\ 1 & 4 & 16 & 64 & 5 \\ 1 & 6 & 36 & 216 & 6 \end{array}\right] \end{array} \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array}$$

Erste Spalte eliminieren:
$\Rightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 4 & 16 & 64 & 5 \\ 0 & 6 & 36 & 216 & 6 \end{array}\right] \begin{array}{c} \\ (-\mathrm{I}) \\ (-\mathrm{I}) \\ (-\mathrm{I}) \end{array}$$

2. Spalte elim. $\Rightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 24 & 192 & 2 \end{array}\right] \begin{array}{l} \\ \\ (-2\,\mathrm{II}) \\ (-3\,\mathrm{II}) \end{array}$$

3. Spalte $\Rightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 0 & 48 & -4 \end{array}\right] \begin{array}{l} \\ \\ \\ (-3\,\mathrm{III}) \end{array}$$

Also hat

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 8 & 48 \\ 0 & 0 & 0 & 48 \end{bmatrix} \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -4 \end{pmatrix}$$

die gleichen Lösungen wie $*$.

Wir machen jetzt Rücksubstitution:

I $\quad 48a = -4 \quad \Rightarrow \quad a = -\frac{4}{48} = -\frac{1}{12}$

II $\quad 8b + 48a = 2 \quad \Rightarrow \quad b = \frac{2-48a}{8} = \frac{6}{8} = \frac{3}{4}$

III $\quad 2c + 4b + 8a = 1 \quad \Rightarrow \quad c = \frac{1-8a-4b}{2} = \frac{1+\frac{2}{3}-3}{2} = -\frac{2}{3}$

IV $\quad d = 1$

Also interpoliert $f(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 + \frac{2}{3}x + 1$

unsere Datenpunkte.