2. Matrix powers (bonus, hand-in) (★☆☆)

For a natural number $k \geq 1$, we define the k-th power of a square matrix A as the matrix multiplication

$$A^k = \underbrace{AA\cdots A}_{k\text{ times}}$$

Moreover we define $A^0 = I$, where I is the identity matrix.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Use induction to show that for any $k \geq 0$ the k-th power of the matrix A is

$$A^k = \begin{bmatrix} 1+k & k \\ -k & 1-k \end{bmatrix}$$

Induktionsverankerung: Für $k=0$ gilt $A^0 = I = \begin{bmatrix} 1+0 & 0 \\ -0 & 1-0 \end{bmatrix}$ ✓

Induktionshypothese: Angenommen, für ein $n \in \mathbb{N}$ gilt:

$$A^n = \begin{bmatrix} 1+n & n \\ -n & 1-n \end{bmatrix}.$$

Induktionsschritt: $A^{n+1} = A^n \cdot A \overset{IH}{=} \begin{bmatrix} 1+n & n \\ -n & 1-n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2(1+n)-n & 1+n \\ -2n-(1-n) & -n \end{bmatrix} = \begin{bmatrix} 1+(n+1) & (n+1) \\ -(n+1) & 1-(n+1) \end{bmatrix}$$

Deshalb stimmt die Aussage auch für $n+1$.

Also gilt die Aussage für alle $k \in \mathbb{N}$.

3. Reconstruct a linear transformation (★☆☆)

a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\2\end{pmatrix}, \; T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}2\\3\\2\end{pmatrix}.$$

Determine the general formula for $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$ with $x, y \in \mathbb{R}$.

b) Find a matrix A such that $T_A = T$.

a) Es gilt $\begin{pmatrix}x\\y\end{pmatrix} = y \cdot \begin{pmatrix}1\\1\end{pmatrix} + (x-y)\begin{pmatrix}1\\0\end{pmatrix}$

Also folgt: $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = T\left(y\begin{pmatrix}1\\1\end{pmatrix} + (x-y)\begin{pmatrix}1\\0\end{pmatrix}\right)$

$$\overset{\text{Linearität von } T}{=} yT\left(\begin{pmatrix}1\\1\end{pmatrix}\right) + (x-y)\,T\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= y\begin{pmatrix}2\\3\\2\end{pmatrix} + (x-y)\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$= \begin{pmatrix}2y + x-y\\ 3y + x-y \\ 2y + (2x-2y)\end{pmatrix} = \begin{pmatrix}x+y\\x+2y\\2x\end{pmatrix}$$

Also wird $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x+y\\x+2y\\2x\end{pmatrix}$

b) Theorem 2.26 $\quad A = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$

$$T(e_1) = T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$T(e_2) = T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}0+1\\0+2\\0\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix}$$

Also gilt $A = \begin{pmatrix}1&1\\1&2\\2&0\end{pmatrix}$.

5. Matrix multiplication (★★★)

a) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find $x, y, z \in \mathbb{R}$ such that $A^3 + xA^2 + yA + zI = 0$. Note that both I and 0 are 3×3 matrices in this equation.

b) Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

We say that a square matrix A is *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the *nilpotent degree* of A.

c) Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A. In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB?

d) Let A be an $m \times m$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I - A)(I + A + \ldots + A^{k-1}) = I$.

e) Let T be an $m \times m$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^m = 0$, i.e. T is nilpotent of degree less or equal to m.

Hint: Even if you do not manage to solve a question, you can use its result to tackle subsequent questions.

c) Nach c) gilt $(AB)^k \overset{b)}{=} A^k B^k = 0 B^k = 0$

Also ist AB nilpotent mit Nilpotenzgrad höchstens k.

Achtung, der Nilpotenzgrad muss nicht gleich k sein!

Gegenbeispiel: Sei A nilpotent mit Grad 5, also $A^5 = 0$.

Sei $B = A^2$. Dann ist $A \cdot B = A^3$ nilpotent mit Grad 2.

d) Wir verwenden Distributivität der Matrixmultiplikation:

$$(I - A)(I + A + \ldots + A^{k-1}) = (I + A + \ldots + A^{k-1}) - A \cdot (I + A + \ldots + A^{k-1})$$

$$= I + A + \ldots + A^{k-1} - A - A^2 - \ldots - A^k$$

$$= I - A^k$$

$$= I \quad \text{(A nilpotent mit Grad k)}$$