1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \ldots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

b) Determine the rank of A. You need to motivate your answer.

a) $A_2 = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ $\qquad A_4 = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{pmatrix}$

$$\begin{array}{cccc} & v_1 & v_2 & v_3 = 2v_2 - v_1 \end{array}$$
$A_3 = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$

$v_1 \quad v_2 \quad v_3 = 2v_2 - v_1 \quad v_4 = 2 \cdot v_3 - v_2 = 3v_2 - 2v_1$

Berechnen, was rank(A_1, A_2, A_3) sind, um Beweisideen zu bekommen

b) $\text{rank}(A_2) = 2, \quad \text{rank}(A_3) = 2, \quad \text{rank}(A_4) = 2$

Vermutung: $\quad \text{rank}(A_m) = 2$

Das ist nur äquivalent zu linear abhängig, weil $v_1 \neq 0$!

1\. $\text{rank}(A_m) \neq 0 \Rightarrow \text{rank}(A) \geq 1$

2\. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \end{pmatrix} \qquad \lambda \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \qquad \begin{array}{l} \lambda 2 = 3 \Rightarrow \lambda = \frac{3}{2} \\ \lambda 3 = 4 \Rightarrow \lambda \frac{4}{3} \end{array} \quad \lightning$

$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \end{pmatrix}$ linear unabhängig $\Rightarrow \text{rank}(A) \geq 2$

3\. Sei $v_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$.

Es gilt $\quad v_2 - v_1 = \begin{pmatrix} 1+2 \\ 2+2 \\ \vdots \\ m+2 \end{pmatrix} - \begin{pmatrix} 1+1 \\ 2+1 \\ \vdots \\ m+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Sei $i > 2$. Es gilt $\quad v_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = \begin{pmatrix} 1+i \\ 2+i \\ \vdots \\ m+i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ m \end{pmatrix} + i \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$= v_1 + i \cdot (v_2 - v_1)$.

Also sind alle v_i, $i > 2$ linear abhängig. Also muss $\text{rank}(A) = 2$.